PROBLEMS FOR THE 11^{th} INTERNATIONAL TOURNAMENT OF YOUNG MATHEMATICIANS

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Contents

Keywords: 1. number theory – 2. games – 3. rational numbers – 4. inequalities – 5. real analysis, dense sequences – 6. algebra, operations – 7. group theory, graphs – 8. graph theory, embeddings – 9. number theory, algorithms – 10. metric spaces, geometry

Notation

$\mathbb{N} = \{1, 2, 3, \dots\}$	set of natural numbers (positive integers)
$\mathbb{Z}_n = \{0, 1, \dots, n-1\}$	ring of residue classes modulo n
$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	sets of integer, rational, real and complex numbers
$\mathbb{R}_+$ and $\mathbb{R}_{\geq 0}$	sets of positive and non-negative real numbers
$\mathbb{R}^2$	real plane
$]a, b[$ and $[a, b]$	open and closed interval from a to b
$\mathbb{Z}[i]$	ring of Gaussian integers $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$
$C^+[0, 1]$	set of strictly positive continuous functions from $[0, 1]$ to $\mathbb{R}$
$M \times N$	Cartesian product of two sets M and N
$\gcd(x_1, \dots, x_n)$	greatest common divisor of $x_1, \dots, x_n$
$\lvert x\rvert$	absolute value of x
$\#M$ or $\lvert M\rvert$	cardinality of a set M
$\operatorname{diam}(M)$	diameter of a set M
$\inf\limits_{x \in M} f(x)$ and $\sup\limits_{x \in M} f(x)$	infimum and supremum of a set $\{f(x) \mid x \in M\}$

1. A Divisibility Problem

1. Let n, p be positive integers such that p is a prime divisor of $2^{2^n} + 1$. Prove that 2^{n+1} divides $p - 1$.

2. We are interested in the triples of positive integers (a, b, c) such that $2^n a + b$ divides $c^n + 1$ for all positive integers n.

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(a) Suppose $a = b = 1$. Is it true that c is necessarily even? That $c = 2^{2k-1}$ for some positive integer k?
(b) Find all triples (a, b, c), satisfying the above condition.

3. Now, suppose that the positive integers a, b, c are such that $2^n a + b$ divides $c^n + 1$ for all integers n of the form $a_k = p^k$, $k \geq 0$, where p is a prime.

(a) For which values of c and p, is it necessarily true that $a = b = 1$?
(b) For a fixed value of p, what can you say about a, b, c, in general?

4. What happens if $2^n a + b$ divides $c^{n+2} + 1$ for all n (or for all $n = p^k$) instead?

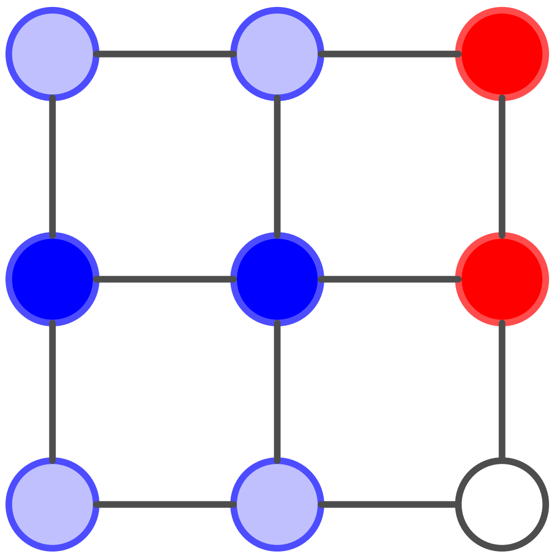
5. Investigate the previous parts for $m^n a + b$ (the above questions correspond to the case $m = 2$) or for other infinite sequences a_k.

6. Suggest and study additional directions of research.

2. **A Graph Coloring Game**

Bob and Riley play a game on a simple connected undirected graph G with $n \geq 2$ vertices. Bob's favourite color is blue and Riley's is red. The goal of each of them is to have as many vertices as possible in their favourite color. Initially all vertices are white.

The players make alternating moves starting with Bob. On his first move, each player chooses any white vertex and colors it in his favorite color. On each consequent move, each player picks a white vertex connected with an edge to a vertex of his color (if there is no such vertex, the player skips his move). Then he colors it in his color. The game finishes when no white vertices remain.



In this example there are two moves made both from Bob And Riley marked with strong blue and red. The possible moves of Bob are marked with opaque blue.

Let $B(G)$ be the maximal number of blue vertices, which Bob can guarantee at the end of the game and $R(G)$ be the maximal number of red vertices, which Riley can guarantee at the end of the game. The strategies with which they achieve this are called optimal.

1. Describe the optimal strategy for Bob and evaluate $B(G)$ where:

(a) G is a tree;
(b) G is a grid graph $M \times K$;
(c) G is a torus grid graph $M \times K$.

2. Let $M_B(n) = \min\limits_{G \in S(n)} B(G)$, where $S(n)$ is the set of all simple undirected connected graphs with n vertices. Provide as best as possible upper and lower bounds for $M_B(n)$.

3. Provide as best as possible upper and lower bounds for $\lim\limits_{n\to\infty} \frac{M_B(n)}{n}$.

For the next 3 problems consider the following modification of the game rules. Instead of choosing which vertices to color, each of the players has a pawn (blue for Bob and red for Riley), which can move from vertex to vertex connected with an edge. Each pawn colors the vertex it is on in its color. The red pawn cannot step on blue vertices and the blue pawn cannot step on red vertices (but the blue pawn can step back on a blue vertex and the red pawn can step back on a red vertex).

Initially Bob places his blue pawn on some vertex and than Riley places his red pawn on a different vertex. Then they take turns to move their pawns. In a turn, the pawn must be moved if possible (it cannot stay in the same vertex, unless there are no possible moves).

Since the game (with the new rules) can continue indefinitely, for this case let $B'(G)$ be the maximal number of blue vertices Bob can guarantee in a finite amount of moves and $R'(G)$ be the maximal number of red vertices Riley can guarantee in a finite amount of moves.

4. Evaluate $B'(G)$ and $R'(G)$ where:
  (a) G is a grid graph $M \times K$;
  (b) G is a torus grid graph $M \times K$.

5. Show that there exist graphs with n vertices such that $B'(G) + R'(G) < n$. And find the minimal n for which such a graph exists.

6. Provide as best as possible upper and lower bounds of $\max\limits_{G \in S(n)} (n - B'(G) - R'(G))$ with regards to n.

7. Suggest and study additional directions of research.

3. **Derivatives of Rational Numbers**

The function $': \mathbb{Q} \to \mathbb{Q}$ satisfies the following properties:

- $p' = 1$ for all positive integer primes p;
- $(mn)' = m'n + mn'$ for all $m, n \in \mathbb{Q}$;
- $0' = 0$.

Also, for any non-negative integer k let $x^{(k)}$ denote k iterations of $'$ on x.

1. (a) Verify that indeed there is a unique such function.
  (b) Prove also that for any $L \in \mathbb{Q}$ there exist $x, y \in \mathbb{Q}$ such that $x' < L < y'$. Is this true for $x'' = x^{(2)}$?

2. Discuss the existence and number (finite or infinite) of solutions to
  (a) $x' = a$ where $a \in \mathbb{Q}$ is given. Start with $a = 0$ and $a = 1$.
  (b) $x' = ax + b$ where $a, b \in \mathbb{Q}$ are given. Start with $b = 0$.
  (c) $x'' = ax' + bx$ where $a, b \in \mathbb{Q}$ are given.
  (d) $x^{(k)} = x$, $x' \neq x$ where $k \geq 2$ is a given positive integer.

Where possible, describe all solutions or generate families of such.

3. Find conditions on x for which the sequence $x^{(k)}$, $k = 0, 1, 2, 3, \ldots$ is monotonic, bounded, convergent (possibly to infinity), etc.

4. (a) Find all $x \in \mathbb{Q}$ such that $x' \equiv 0 \pmod 4$ (that is, in lowest terms its denominator is odd and its numerator is divisible by 4).

(b) What about $x' \equiv 0 \pmod{p}$ where p is a given odd prime (that is, in lowest terms its numerator is divisible by p and its denominator is not)?

5. Investigate the above problems for other sets (e.g. $\mathbb{Q}(i)$) or other similar functions (e.g. the logarithmic derivative $\frac{x'}{x}$).

6. Suggest and study additional directions of research.

4. Inequalities with a Shift

Let $x_1, x_2, \ldots, x_n$ be positive real numbers, and let k be a positive integer. Define $x_{n+1} = x_1$, $x_{n+2} = x_2, \ldots$

1. Denote by $\sigma_p(n)$ the cyclic shift of numbers 1 to n by $p-1$ positions. For example, $\sigma_3(1) = 3$, $\sigma_3(2) = 4, \ldots, \sigma_3(n-2) = n$, $\sigma_3(n-1) = 1$, $\sigma_3(n) = 2$.

The goal of this item is to investigate for which cyclic shifts $\sigma_p(n)$ the inequality

$$\sum_{i=1}^{n} \left(\frac{x_i}{x_{i+1}}\right)^k \geq \sum_{i=1}^{n} \frac{x_i}{x_{\sigma_p(i)}} \tag{1}$$

is satisfied for all positive numbers $x_1, x_2, \ldots, x_n$.

(a) Prove (1) for $p = 1$.
(b) Prove (1) for $p = k + 1$.
(c) Prove (1) for $k = p = n$.
(d) Prove (1) for $1 < p < k + 1$.
(e) For which $k + 1 < p < n$ the inequality (1) is true?

2. Suppose now k is an arbitrary real number. For which p (depending on k) does the inequality (1) hold for all positive numbers $x_1, x_2, \ldots, x_n$?

3. Assume $k = 2$, $n = 3$. Find all permutations σ of the numbers from 1 to n such that the following inequality:

$$\sum_{i=1}^{n} \left(\frac{x_i}{x_{i+1}}\right)^k \geq \sum_{i=1}^{n} \frac{x_i}{x_{\sigma(i)}} \tag{2}$$

is true for all positive $x_1, x_2, \ldots, x_n$.

4. Assume now that $k = 2$, $n > 3$. Find all permutations, for which the inequality (2) holds.

5. Investigate further directions and generalisations of the problem. For example, the left hand side of the inequalities can be replaced by

$$\sum_{i=1}^{n} \left(\frac{x_i}{x_{i+1} + x_{i+2}}\right)^k, \quad \text{or} \quad \sum_{i=1}^{n} \left(\frac{x_i + x_{i+1}}{x_{i+2} + x_{i+3}}\right)^k, \quad \text{or} \quad \sum_{i=1}^{n} \left(\frac{x_i + x_{i+1}}{x_{i+1} + x_{i+2}}\right)^k.$$

5. Dense Sets of Fractions

Let $x = \{x_n\}_{n=1}^{\infty}$ be a strictly monotonous and unbounded from above sequence of positive real numbers. We say that $x = \{x_n\}_{n=1}^{\infty}$ is *of type 0* and write $x \in T_0$, if its set of fractions $\theta_x = \{x_n/x_m \mid n, m \in \mathbb{N}\}$ is dense in the open interval $]0, \infty[$. This means that any non-empty interval $(\alpha, \beta) \subset]0, \infty[$ should contain at least one point $t \in \theta_x$.

Starting from the sequence $x = \{x_n\}_{n=1}^{\infty}$, define the sequence of its partial sums:

$$S(x) = \{x_1 + \cdots + x_n\}_{n=1}^{\infty}.$$

We shall say that $x = \{x_n\}_{n=1}^{\infty}$ is *of type* $m \in \mathbb{N}$ and write $x \in T_m$ if $S(x) \in T_{m-1}$.

1. (a) Let $x = \{x_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}$ be the sequence of all natural numbers. Determine, for which $m \in \mathbb{N} \cup \{0\}$ we have $x \in T_m$.
(b) Investigate the same question when $x = \{x_n\}_{n=1}^{\infty}$ is an arithmetic or geometric progression.

2. Do there exist sequences $x = \{x_n\}_{n=1}^{\infty}$ such that:

(a) $x \in T_1$, but $x \notin T_0$;
(b) $x \in T_0$, but $x \notin T_1$;
(c) $x \in T_m$, but $x \notin T_l$ for some $l \neq m$?

3. Investigate how the condition $x \in T_0$ is related to other properties of the sequence $x = \{x_n\}_{n=1}^{\infty}$. In particular, if $y = \{y_n\}_{n=1}^{\infty} \subset]0, \infty[$ is another sequence, we write $x_k \approx y_k$ or $x \approx y$, if $\lim_{k\to\infty} x_k/y_k = 1$. Let also $\pi_x(n) = \max\{k \mid x_k < n\}$.

Which of the following conditions imply that $x \in T_0$ and which of them are implied by $x \in T_0$:

(a) There exists $y \in T_0$ such that $x \approx y$.
(b) There exists $y \in T_0$ such that $\pi_x(n) \approx \pi_y(n)$.
(c) There exist $y, z \in T_0$ such that $x = y + z$.
(d) We have $x_{n+1} \approx x_n$, that is $\lim_{k\to\infty} x_{k+1}/x_k = 1$.
(e) Starting from some number n we have $\frac{9}{10} < \frac{\pi_x(n)}{n/\ln n} < \frac{11}{10}$.
(f) $x = \{x_n\}_{n=1}^{\infty}$ is the sequence of all prime numbers; or the sequence of all prime numbers which belong to some arithmetic progression starting with 1 and having a rational difference.
(g) $x = \{x_n\}_{n=1}^{\infty}$ is a sequence of all primes in the sequence of integral parts of an arithmetic sequence with irrational difference.

Investigate the similar questions for the condition $x \in T_m$, $m \geq 1$.

4. Consider the set $C^+[0, 1]$ of all strictly positive continuous functions on the closed interval $[0, 1]$. For any $f, g \in C^+[0, 1]$ we say that $f < g$ if $f(x) < g(x)$ for all $x \in [0; 1]$. Define an interval (f, g) as a set $\{h \in C^+[0, 1] \mid f < h < g\}$. Finally, a subset $\theta \subset C^+[0, 1]$ will be called dense, if any non-empty interval (f, g) contains at least one element from θ.

Let $F = \{f_n\}_{n=1}^{\infty} \subset C^+[0, 1]$ be a strictly increasing sequence of strictly increasing functions, that is each function $f_k(x)$ strongly increases on the interval $[0, 1]$ and for any $k \in \mathbb{N}$ we have $f_k < f_{k+1}$. We say that $F = \{f_n\}_{n=1}^{\infty}$ is of type 0 and write $F \in T_0$, if the set of fractions $\theta_F = \{f_k/f_l \mid k, l \in \mathbb{N}\}$ is dense in $C^+[0, 1]$. Is it true that such sequences F exist?

6. **Operation Tables**

Let $q \geq 2$ be an integer. Denote by $[\![0, q-1]\!]$ the set of integers from 0 to $q - 1$. A function

$$\star\colon [\![0, q-1]\!] \times [\![0, q-1]\!] \to [\![0, q-1]\!]$$

is called a *q-operation*, if there exists an element $e \in [\![0, q-1]\!]$ such that:

- for all $x \in [\![0, q-1]\!]$ we have $\star(x, e) = x$;
- for any $x, y \in [\![0, q-1]\!]$ we have $\star(e, \star(y, x)) = \star(x, y)$.

If $\star$ is a q-operation, we shall write $x \star y$ instead of $\star(x, y)$.

For any function $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ we can construct a function

$$f_q\colon [\![0, q-1]\!] \times [\![0, q-1]\!] \to [\![0, q-1]\!], \quad (x, y) \mapsto f(x, y) \mod q.$$

For example, for $f(x, y) = x - y$ and $q = 10$ we have $f_q(2, 5) = 7$, since $2 - 5 = -3 \equiv 7 \mod 10$. Fir simplicity, we shall use the notation $+_q$, $-_q$, $\times_q$, $\gcd_q$ for functions $x + y$, $x - y$, $x \times y$, $\gcd(x, y)$.

1. Verify that $+_q$, $-_q$, $\gcd_q$ and $\times_q$ are q-operations and find the corresponding element e for each of them. Can you find other examples of q-operations?

2. Find the number of possible q-operations as a function of q.

We say that a subset $D \subset [\![0, q-1]\!]$ is $\star$-*complete*, if for any $z \in [\![0, q-1]\!]$ there exist elements $x, y \in D$ such that $x \star y = z$. Define

$$c_q(\star) = \min\{\#D, D \text{ is } \star\text{-complete}\}.$$

3. Estimate $c_q(\gcd_q)$, $c_q(-_q)$, $c_q(+_q)$ and $c_q(\times_q)$ as functions of q.

4. What can you say about $c_{q+1}(\star_{q+1}) - c_q(\star_q)$ for $\star \in \{\gcd, +, -, \times\}$?

For any q-operation $\star$ consider the number of smallest $\star$-complete subsets:

$$d_q(\star) = \#\{D \subset [\![0, q-1]\!], D \text{ is } \star\text{-complete and } \#D = c_q(\star)\}.$$

5. Study the numbers $d_q(\star)$ for $\star \in \{\gcd, +, -, \times\}$. In particular, try answering the following questions:

(a) When q divides $d_q(\star)$?
(b) Can you give an estimate for $d_q(\star)$?
(c) Are there subsequences of $d_q(\star)$ that form (infinite) arithmetic sequences?

6. Investigate other interesting operations $\star$ or explore other tracks of research.

7. **Graphs of Finite Groups**

For any finite group G define an undirected graph $\Gamma(G)$ as follows. We set $V(\Gamma(G)) = G \setminus \{1\}$ and connect two vertices x and y by an edge, if there exist natural numbers n and m with $x^n = y^m \neq 1$.

Let S_n and A_n be the symmetric and the alternating groups on n elements respectively. The *dihedral group* D_n of order $2n$ can be defined as $\langle x, y \,|\, x^n = y^2 = (xy)^2 = 1\rangle$ or, equivalently, as the symmetry group of a regular n-gon. Finally, by $\mathbb{Z}_n$ in this problem we understand the additive group of the set $\mathbb{Z}_n$.

For example, the graph $\Gamma(D_3) = \Gamma(S_3)$ looks as follows:

Here the two connected vertices correspond to the rotations of the regular triangle by angles $\pm 2\pi/3$, and the remaining three isolated vertices correspond to the three reflection symmetries of the triangle.

1. Draw $\Gamma(\mathbb{Z}_4)$, $\Gamma(D_4)$, $\Gamma(A_4)$ and $\Gamma(S_4)$.

2. For a natural n compute the number of edges in the following graphs: $\Gamma(\mathbb{Z}_n)$, $\Gamma(D_n)$, $\Gamma(A_n)$ and $\Gamma(S_n)$.

3. Which of the graphs from the previous question are not connected? Try to find or estimate the number of connected components for them.

4. Which of the graphs from the question 2 (or their connected components, if the graphs are not connected)

(a) have Eulerian circuit?
(b) have Hamiltonian circuit?
(c) are planar?

(a) have Eulerian circuit?
(b) have Hamiltonian circuit?
(c) are planar?

5. Is it true that $\Gamma(Z_p) \simeq K_{p-1}$ for prime p, where K_n is a complete graph with n vertices?

6. Describe

(a) $\Gamma(\mathbb{Z}_n)$ for any $n \geq 2$;
(b) $\Gamma(D_n)$ for any $n \geq 2$;
(c) $\Gamma(A)$ for an arbitrary finite abelian group A.

7. Study $\Gamma(G)$ for other finite groups such as a general linear group $GL(n, \mathbb{Z}_p)$ (at least for some small n and p) or the symmetries of an n-dimensional hypercube?

8. **Embeddings of Connected Graphs**

Throughout let d be a (finite) positive integer and G be a connected undirected simple graph. An *embedding* of G in $\mathbb{R}^d$ is a map associating the vertices of G with points in $\mathbb{R}^d$ and the edges of G with simple arcs (homeomorphic images of $[0,1]$) so that: for any edge e, the endpoints of the arc associated with e are the points associated with the end-vertices of e.

Denote by $\mathcal{E}_d(G)$ the set of all embeddings of G in $\mathbb{R}^d$. For vertices u, v of G we define the distance $d_G(u,v)$ to be the number of edges in a shortest path from u to v. Treating $||\cdot||$ as the standard Euclidean norm in $\mathbb{R}^d$, define the *distortion* of G as

$$D_d(G) = \inf_{f \in \mathcal{E}_d(G)} \frac{\max_{u,v \in G, u \neq v} \frac{||f(u)-f(v)||}{d_G(u,v)}}{\min_{u,v \in G, u \neq v} \frac{||f(u)-f(v)||}{d_G(u,v)}}$$

1. For $d = 2$ compute (or give bounds on) the distortion of the following graphs:

(a) C_n – a simple cycle graph of length n;
(b) L_n – the ladder graph with vertex set $\{v_1, \ldots, v_n, w_1, \ldots, w_n\}$ and edge set $\{v_i w_i : 1 \leq i \leq n\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{w_i w_{i+1} : 1 \leq i \leq n-1\}$;
(c) Q_n – the hypercube graph formed from the vertices and edges of an n-dimensional hypercube;
(d) an arbitrary planar graph.

In case you give general bounds, you can check which particular graphs of the corresponding type attain them.

2. Now let G be a tree. We say that G is *perfect k-ary* if there is a unique vertex r (called the *root*) of degree k such that the distance $d_G(r,l)$ is the same for any leaf l, and each vertex, except the root and the leaves, has degree $k+1$.

(a) Prove that there exist constants c_1, c_2 and an integer $N \in \mathbb{N}$ such that for every *perfect binary* (2-*ary*) tree G on $n \geq N$ vertices we have $c_1 \leq \frac{D_1(G)\log_2 n}{n} \leq c_2$. Estimate c_1 and c_2.
(b) Can one derive a similar statement for $d \geq 2$?
(c) Consider the above for perfect $k-$ary trees, $k \geq 3$.
(d) Compute (or give bounds on) distortions for other particular trees and in the general case. In case you give general bounds, you can check which particular graphs of the corresponding type attain them.

3. Consider distortions of other connected graphs. Suggest and study additional directions of research.

9. **Maximal Orders of Residues**

Let b and n be two integers such that $\gcd(b, n) = 1$. Let d be the minimal positive integer such that $b^d \equiv 1 \pmod{n}$. We will say that d *is the order of* b *modulo* n, e.g. the order of 2 modulo 7 equals 3. Let $f(n)$ be the maximal positive integer k such that there exists a number b of order k modulo n. A number b of order $f(n)$ modulo n will be called *an element of maximal order modulo* n. Denote by $g(n)$ the number of distinct elements of maximal order modulo n (elements a and b are distinct modulo n if $a \not\equiv b \pmod{n}$). For example, $f(5) = 4$, $g(5) = 2$ and the integers 2 and 3 are the elements of maximal order modulo 5.

1. (a) Prove that $f(n)$ is finite.
(b) For small n $(2 \leq n \leq 10)$ calculate $f(n)$ and $g(n)$ and find all elements of maximal order modulo n.
(c) Prove that $f(p) = p - 1$, where p is a prime. What is the value of $g(p)$? Does there exist a composite n such that $f(n) = n - 1$?
(d) Investigate the previous question for $n = p^\alpha$, $\alpha \in \mathbb{N}$, p is a prime odd number. Suggest an efficient (polynomial) algorithm that finds an element of maximal order modulo n, in case that some element of maximal order modulo p is known. Try to calculate $g(n)$ for such n.
(e) Try to estimate $f(n)$ and $g(n)$ for arbitrary n. Suggest algorithms for finding an element of maximal order modulo n.

2. Consider the set $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$, where i is the imaginary unit. We say that $n \in \mathbb{Z}[i]$ is *divisible* by $m \in \mathbb{Z}[i]$ if there exists $q \in \mathbb{Z}[i]$ such that $n = qm$. For $c, d \in \mathbb{Z}[i]$ and $n \in \mathbb{Z}[i] \setminus \{0\}$, we write $c \equiv d \pmod{n}$ if $c - d$ is divisible by n. Analogously, one can define functions $f(n)$, $g(n)$ and elements of maximal order modulo n. Investigate analogues of the questions 1(a) – 1(e).

3. Consider the set $\mathbb{Z}[\sqrt{2}] = \left\{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\right\}$. Define arithmetic and modular operations on $\mathbb{Z}[\sqrt{2}]$ and investigate the questions 1(a) – 1(e) for $\mathbb{Z}[\sqrt{2}]$.

4. Investigate the same problem for the set $\mathbb{Z}[\sqrt[3]{2}] = \left\{a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in \mathbb{Z}\right\}$.

10. **Optical Illusions**

Let (M, d) and (N, q) be two metric spaces with distance functions d, q. We say that they are *visually equivalent* if there exists a function $f : M \longrightarrow N$ called *visual equivalence* with the following properties:

- there exist $K > 0$ and $C > 0$ such that $\frac{1}{K}d(x, y) - C \leq q(f(x), f(y)) \leq Kd(x, y) + C$ for all $x, y \in M$;
- there exists $R > 0$ such that for any $a \in N$ there is $b \in f(M)$ for which $q(a, b) \leq R$.

1. Given an undirected simple connected possibly infinite graph $\Gamma = (V, E)$, we introduce a metric space structure on V as follows:

$$d_\Gamma(u, v) = \min\{n \in \mathbb{N} \mid \text{there is an edge path of length } n \text{ connecting } u \text{ and } v \text{ in } \Gamma\}$$

provided that length between adjacent vertices is 1. A function $\gamma : \mathbb{N} \longrightarrow V$ is said to be a *ray* if $d_\Gamma(\gamma(x), \gamma(y)) = |x - y|$ for all $x, y \in \mathbb{N}$. Determine which of the following properties of graphs are stable under visually equivalence:

– to have a finite diameter $\operatorname{diam}(M) = \sup_{x,y \in M} d_\Gamma(x, y)$;
– to have exactly one ray starting at some fixed vertex;
– to have exactly two rays starting at some fixed vertex.

2. Determine which of the following metric spaces (with standard distance functions) are visually equivalent:

(a) $\mathbb{R}$, $\mathbb{Z}$, $[0, 1]$;
(b) $\mathbb{R}$, $\mathbb{R}_{\geq 0}$;
(c) $\mathbb{R}^2$, $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, $\mathbb{R} \times \mathbb{R}_{\geq 0}$;
(d) $\mathbb{R}^2$, $\{(x, y) \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0\}$.

3. Does there exist:

(a) a graph that is visually equivalent to $\mathbb{R}^2$?
(b) a tree (a connected graph with no cycles) that is visually equivalent to $\mathbb{R}^2$?

4. For which $n \geq 2$ an infinite tree with one vertex of degree n and all other vertices of degree 2 is visually equivalent to some subset of $\mathbb{R}^2$? Study the case of a tree with more than one big degree vertex.

5. Suggest and investigate your own directions of this problem.

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